JOURNAL OF THERMOPHYSICS AND HEAT TRANSFER
Vol. 20, No. 4, October–December 2006

Technical Notes

TECHNICAL NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Generalizing the Method of Kulish to One-Dimensional Unsteady Heat Conducting Slabs

J. I. Frankel*
University of Tennessee, Knoxville, Tennessee 37996-2210

DOI: 10.2514/1.22995

Nomenclature

a_j	=	values of (0 or 1), $j = 1, 2$
b	=	slab width, m
b_j	=	values of (0 or 1), $j = 1, 2$
C	=	heat capacity, kJ/(kg°C)
F	=	arbitrary heat source, °C/m^2
$f_{\max}$	=	forcing function, °C/m^2
G	=	Green's function
$g(x)$	=	initial condition, °C
k	=	thermal conductivity, W/(m°C)
q''	=	dimensional heat flux, W/m^2
s	=	dummy time variable, s
T	=	temperature, °C
T_o	=	initial temperature, °C
t	=	time, s
t_o	=	dummy time variable, s
u	=	dummy time variable, s
w_x	=	weight function
x	=	spatial variable, m
x_o	=	dummy spatial variable, m
α	=	thermal diffusivity [k/(ρC)], m^2/s
ϵ	=	small increment, s
η	=	fixed position, m
ρ	=	density, kg/m^3

Introduction and Background

A finite region generalization of the method of Kulish is developed applicable to transient, one-dimensional heat conduction in a slab geometry under the assumptions of constant thermophysical properties and the existence of an arbitrary volumetric heating source. A novel relationship is developed with the aid of the Green's functions method and regularization as associated with weakly singular integral equations. The transient, one-dimensional heat equation is used to convey the mathematical approach and its significance to the experimental community. In the limit to the half-space with known heat source, the integral relationship permits the local heat flux to be predicted using a single thermocouple and thus provides a simple and cheap means for obtaining in-depth heat fluxes. These data require simple digital filtering in order to stabilize the weakly, ill-posed nature of the formulation. It also casts interest into the direct measurement of heating/cooling rate, $\mathrm{d}T/\mathrm{d}t$ as previously suggested by the author and his colleagues [1–8].

Kulish and his colleagues [9–11] recently suggested an elegant integral relationship between temperature and heat flux for an unsteady, linearly conducting medium in the half-space geometry. The unsteady, half-space heat conduction equation investigated was given as

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}(x,t) = \frac{\partial^2 T}{\partial x^2}(x,t), \qquad (x,t) > 0 \tag{1a}$$

subject to the initial condition

$$T(x,0) = T_o, \qquad x \geq 0 \tag{1b}$$

The temperature-flux integral relationship was derived through the use of Laplace transforms and formally leads to the result

$$T(x,t) = T_o + \frac{1}{\sqrt{\rho C k \pi}}\int_{u=0}^{t} q''(x,u)\frac{\mathrm{d}u}{\sqrt{t-u}}, \qquad (x,t) \geq 0 \tag{1c}$$

This equation can be viewed as a classical Abel integral equation [3–5,12–15] for the heat flux, $q''(x,t)$ in time for fixed position. Regularizing (in the sense of singular integral equations, that is, iterated kernels) leads to the inversion

$$q''(x,t) = \sqrt{\frac{\rho C k}{\pi}}\int_{u=0}^{t}\frac{\partial T}{\partial u}(x,u)\frac{\mathrm{d}u}{\sqrt{t-u}}, \qquad (x,t) \geq 0 \tag{1d}$$

Kulish et al. [9] have also developed a heat flux relationship in the half-space contained in the presence of a volumetric heating source. This contribution offers a new mathematical formalism that allows for the study of the finite width problem.

Some mathematical observations are now related concerning Eqs. (1c) and (1d) with regard to specifying T or q'' in the presence of discrete, noisy data sets. First, given the heat flux, $q''(x,t)$ in Eq. (1c), the temperature can be determined by direct numerical calculation. This weakly singular integral poses no numerical difficulties nor is this calculation ill posed. Second, if provided discrete noisy temperature data, Eq. (1c) now represents an Abel integral equation which is mildly ill posed [5,12–15]. That is, the root-mean square error of the heat flux grows as the sample density increases. In the presence of white noise [5], this error grows as the square root of the sample density, $\sqrt{M}$, where M is the total number of data in the set. Numerical calculations for the heat flux are unstable to input perturbations in the temperature data and hence is ill posed (with respect to temperature data). The inverted equation presented in Eq. (1d) provides insight into this dilemma. The heat flux, $q''(x,t)$ in Eq. (1d) is reconstructed from a weakly singular integral involving the heating/cooling rate, $(\partial T/\partial t)(x,t)$. That is, the error in the measurement of temperature is differentiated and thus explains the difficulty. However, if one could 1) control the high frequency components in the temperature signal through digital filtering or 2) directly measure heating/cooling rate, $(\partial T/\partial t)(x,t)$, then the ill posedness can be controlled. In fact, if a heating/cooling rate,

Received 5 February 2006; revision received 22 May 2006; accepted for publication 23 May 2006.

*Professor, Mechanical, Aerospace and Biomedical Engineering Department. Associate Fellow AIAA.

$(\partial T/\partial t)(x,t)$ sensor is used in the presence of white noise, it can be shown via discrete Fourier transforms (DFT) analysis that the root-mean square of the heat flux error decreases as $\sqrt{\ell n(M)/M}$ as $M \to \infty$. It should be finally noted that numerical differentiation of noisy data represents the most basic inverse problem [16,17].

Before proceeding further, a brief discussion on these equations as a utility for extracting experimentally determined data is warranted. Gardon heat flux gauges [18–21] are based on the particular law associated with infinite thermal speed heat conduction, namely Fourier's law [22]. This law was historically developed based on steady-state observations. Meanwhile, the integral relationships (not solution) displayed in Eqs. (1c) and (1d) involve the coupling between the general and particular laws of heat conduction theory. Thermocouples are relatively cheap, easy to install, and robust. Orientation issues must also be considered in the placement process in order to assure that the thermocouple's temperature is indeed that of the temperature at the local position. This is especially crucial when the thermophysical properties of the thermocouple and material are dissimilar.

Fundamental Results from This Note

The major result from this note is presented upfront in order not to cloud the technical significance of the integral relationships because the content of this note is mathematical in nature. To this end, consider the heat equation in a slab geometry, defined in $x \in [0,\eta]$, $(y,z) \in (-\infty,\infty)$ having constant thermophysical properties (k,ρ,C). Additionally, let an arbitrary heat source, denoted by $F[T(x,t)]$, exist in this slab. The heat equation is given as

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}(x,t) = \frac{\partial^2 T}{\partial x^2}(x,t) + F[T(x,t)], \qquad x \in (0,\eta), \qquad t \geq 0 \tag{2a}$$

subject to the trivial initial condition (without loss of generality)

$$T(x,0) = 0, \qquad x \in [0,\eta] \tag{2b}$$

It will be demonstrated that the heat flux, $q''(x,t)$ can be expressed by the integral relationship

$$\begin{aligned} q''(x,t) = {} & \frac{k}{\sqrt{\alpha\pi}}\Bigg(\int_{u=0}^{t}\frac{\partial T}{\partial u}(x,u)\frac{\mathrm{d}u}{\sqrt{t-u}} \\ & - \int_{u=0}^{t} T(\eta,u)\left[\frac{(x-\eta)^2}{4\alpha}\frac{e^{-(x-\eta)^2/4\alpha(t-u)}}{(t-u)^{\frac{5}{2}}} - \frac{1}{2}\frac{e^{-(x-\eta)^2/4\alpha(t-u)}}{(t-u)^{\frac{3}{2}}}\right]\mathrm{d}u \\ & + \frac{\eta-x}{2k}\int_{u=0}^{t} q''(\eta,u)\frac{e^{-(x-\eta)^2/4\alpha(t-u)}}{(t-u)^{\frac{3}{2}}}\mathrm{d}u\Bigg) \\ & + \frac{k}{2\sqrt{\alpha\pi}}\int_{u=0}^{t}\int_{x_o=x}^{\eta} F[T(x_o,u)](x-x_o)\frac{e^{-(x-x_o)^2/4\alpha(t-u)}}{(t-u)^{\frac{3}{2}}}\mathrm{d}x_o\,\mathrm{d}u, \\ & x \in (0,\eta), \qquad t \geq 0 \end{aligned} \tag{2c}$$

The implications and impact of this relationship have significance to experimental investigations. That is, an analytic expression is now available for the interior region that does not require knowledge of the boundary at $x = 0$. In situations where the boundary condition at $x = 0$ is difficult to obtain, Eq. (2c) permits use of an embedded sensor to obtain the interior heat flux at the temperature sensor location. As with the Green's function formulation, the boundary condition at $x = \eta$ is required. If no source is present and for early-time analysis where η is sufficiently large, then the interior heat flux at the sensor location is determined from a mere temperature or heating/cooling rate measurement history.

Consider a thermocouple located in depth $x \in (0,\eta)$ with lead wires oriented parallel to the isotherms in order to minimize heat conduction losses within the leads. Additionally, let a known volumetric source function, F be present (which includes $F = 0$). Observe, the integral terms containing both $T(\eta,u)$ and $q''(\eta,u)$ vanish in the limit as $\eta \to \infty$. Measuring the time history of the temperature and performing simple digital filtering can lead to an accurate representation of the heat flux at any depth within the slab (under the imposed noted assumptions). Diffusion naturally damps high frequency information. Thus, digital filtering should be advocated and implemented [5] in order to remove the unnecessary high-frequency components in the signal that play havoc on numerical differentiation. Signal-to-noise issues should be considered when designing the digital filter.

This paper is divided into two additional sections for convenience to the reader. The next section derives Eq. (2c) under the imposed assumptions. This is in contrast to the half-space problems described in the Kulish papers using the Laplace transform method. The Laplace transform method provides limited physical insight for generalization. The fundamental mathematical framework for generalizing this relationship lies in the Green's function [22–26]. The next section presents two example problems validating the analysis.

Derivation of Integral Relationship Based on the Green's Function Method

The boundary element method involves the classical Green's function approach but is based on the full-space Green's function (GF) associated with the differential operator [26]. The GF method begins by operating on the heat equation given in Eq. (2a) with the Green's function denoted as $G(x,t/x_o,t_o)$ and integrating over the domains of interest. Here, G(effect/cause) notation is used [24]. Doing so, we obtain

$$\begin{aligned} & \lim_{\epsilon\to 0}\int_{t_o=0}^{t+\epsilon}\int_{x_o=0}^{\eta}\left[\frac{1}{\alpha}\frac{\partial T}{\partial t_o}(x_o,t_o) - \frac{\partial^2 T}{\partial x_o^2}(x_o,t_o)\right]G(x,t/x_o,t_o)\,\mathrm{d}x_o\,\mathrm{d}t_o \\ & \quad = \lim_{\epsilon\to 0}\int_{t_o=0}^{t+\epsilon}\int_{x_o=0}^{\eta} F[T(x_o,t_o)]G(x,t/x_o,t_o)\,\mathrm{d}x_o\,\mathrm{d}t_o, \\ & \quad x \in (0,\eta), \qquad t > 0 \end{aligned} \tag{3}$$

where causality is noted through the limit displayed in Eq. (3). Integrating by parts, making use of causality and incorporating the trivial initial condition reduces Eq. (3) to

$$\begin{aligned} w_x T(x,t) = {} & \int_{t_o=0}^{t}\bigg[G(x,t/\eta,t_o)\frac{\partial T}{\partial x_o}(\eta,t_o) \\ & - G(x,t/0,t_o)\frac{\partial T}{\partial x_o}(0,t_o) - \frac{\partial G}{\partial x_o}(x,t/\eta,t_o)T(\eta,t_o) \\ & + \frac{\partial G}{\partial x_o}(x,t/0,t_o)T(0,t_o)\bigg]\mathrm{d}t_o \\ & + \int_{t_o=0}^{t}\int_{x_o=0}^{\eta} G(x,t/x_o,t_o)F[T(x_o,t_o)]\,\mathrm{d}x_o\,\mathrm{d}t_o, \\ & x \in [0,\eta], \qquad t \geq 0 \end{aligned} \tag{4a}$$

where w_x is a weight function and the Green's function [23] is determined from the solution of (backward time equation, that is, the adjoint operator [24])

$$\frac{1}{\alpha}\frac{\partial G}{\partial t_o} + \frac{\partial^2 G}{\partial x_o^2} = -\delta(x_o - x)\delta(t_o - t), \qquad (x,x_o) \in (-\infty,\infty) \qquad t_o > t \tag{4b}$$

subject to regularity conditions at $\pm$ infinity and causality. Equation (4b) defines the full-space Green's function and thus requires the introduction of w_x in Eq. (4a). At $x = 0$ and η, $w_x = 0.5$ whereas in the interior the weight function is unity, that is, $w_x = 1$, $x \in (0,\eta)$. The full-space Green's function is given by [23]

$$G(x,t/x_o,t_o) = \sqrt{\frac{\alpha}{4\pi(t-t_o)}}e^{-(x-x_o)^2/4\alpha(t-t_o)} \qquad (x,x_o) \in (-\infty,\infty), \qquad t > t_o \tag{4c}$$

The development of the Green's function in the causal variable is indicated here though most books develop the Green's function in the effect variable [and thus the adjoint operator displayed in Eq. (4b) is replaced by the linear heat operator shown in Eq. (2) based on reciprocity [22]]. The proposed formulation of the Green's function is taken from a mathematical viewpoint. Either approach renders identical results.

The local heat flux, $q''(x,t) = -k\frac{\partial T}{\partial x}(x,t)$ is obtained by operating on Eq. (4a) with $-k\frac{\partial}{\partial x}$. Doing so yields

$$\begin{aligned} w_x q''(x,t) = &\int_{t_o=0}^{t}\bigg[\frac{\partial G}{\partial x}(x,t/\eta,t_o)q''(\eta,t_o) \\ &-\frac{\partial G}{\partial x}(x,t/0,t_o)q''(0,t_o)+k\frac{\partial^2 G}{\partial x\partial x_o}(x,t/\eta,t_o)T(\eta,t_o) \\ &-k\frac{\partial^2 G}{\partial x\partial x_o}(x,t/0,t_o)T(0,t_o)\bigg]\mathrm{d}t_o - k\int_{t_o=0}^{t}\int_{x_o=0}^{\eta}\frac{\partial G}{\partial x}(x,t/x_o,t_o) \\ &\times F[T(x_o,t_o)]\,\mathrm{d}x_o\,\mathrm{d}t_o, \qquad x\in[0,\eta], \qquad t\ge 0 \end{aligned} \tag{5}$$

To begin, let us explicitly express the temperature solution displayed in Eq. (4a) as

$$\begin{aligned} w_x T(x,t) = &\sqrt{\frac{\alpha}{4\pi}}\int_{t_o=0}^{t}\bigg[-\frac{e^{-(x-\eta)^2/[4\alpha(t-t_o)]}}{k\sqrt{t-t_o}}q''(\eta,t_o) \\ &+\frac{e^{-x^2/[4\alpha(t-t_o)]}}{k\sqrt{t-t_o}}q''(0,t_o)-\frac{(x-\eta)}{2\alpha}\frac{e^{-(x-\eta)^2/[4\alpha(t-t_o)]}}{(t-t_o)^{\frac{3}{2}}}T(\eta,t_o) \\ &+\frac{x}{2\alpha}\frac{e^{-x^2/[4\alpha(t-t_o)]}}{(t-t_o)^{\frac{3}{2}}}T(0,t_o)\bigg]\mathrm{d}t_o \\ &+\sqrt{\frac{\alpha}{4\pi}}\int_{t_o=0}^{t}\int_{x_o=0}^{\eta}\frac{e^{-(x-x_o)^2/[4\alpha(t-t_o)]}}{\sqrt{t-t_o}} \\ &\times F[T(x_o,t_o)]\,\mathrm{d}x_o\,\mathrm{d}t_o \qquad x\in[0,\eta], \qquad t\ge 0 \end{aligned} \tag{6}$$

and the heat flux given in Eq. (5) as

$$\begin{aligned} w_x q''(x,t) = &\frac{1}{4\sqrt{\alpha\pi}}\int_{t_o=0}^{t}\bigg[-(x-\eta)\frac{e^{-(x-\eta)^2/[4\alpha(t-t_o)]}}{(t-t_o)^{\frac{3}{2}}}q''(\eta,t_o) \\ &+x\frac{e^{-x^2/[4\alpha(t-t_o)]}}{(t-t_o)^{\frac{3}{2}}}q''(0,t_o)+k\bigg(\frac{e^{-(x-\eta)^2/[4\alpha(t-t_o)]}}{(t-t_o)^{\frac{3}{2}}} \\ &-\frac{(x-\eta)^2}{2\alpha}\frac{e^{-(x-\eta)^2/[4\alpha(t-t_o)]}}{(t-t_o)^{\frac{5}{2}}}\bigg)T(\eta,t_o)-k\bigg(\frac{e^{-x^2/[4\alpha(t-t_o)]}}{(t-t_o)^{\frac{3}{2}}} \\ &-\frac{x^2}{2\alpha}\frac{e^{-x^2/[4\alpha(t-t_o)]}}{(t-t_o)^{\frac{5}{2}}}\bigg)T(0,t_o)\bigg]\mathrm{d}t_o+\frac{k}{4\sqrt{\alpha\pi}} \\ &\times\int_{t_o=0}^{t}\int_{x_o=0}^{\eta}(x-x_o)\frac{e^{-(x-x_o)^2/[4\alpha(t-t_o)]}}{(t-t_o)^{\frac{3}{2}}} \\ &\times F[T(x_o,t_o)]\,\mathrm{d}x_o\,\mathrm{d}t_o \qquad x\in[0,\eta], \qquad t\ge 0 \end{aligned} \tag{7}$$

For notational convenience, let $t_o \to u$ in Eq. (6) and then we begin the interior $[x\in(0,\eta)]$ regularization process by letting $t\to s$ and operate on the resulting expression with $\mathrm{d}s/\sqrt{t-s}$ followed by integration over the domain of interest to get

$$\begin{aligned} &\int_{s=0}^{t}\frac{T(x,s)}{\sqrt{t-s}}\,\mathrm{d}s \\ &=\sqrt{\frac{\alpha}{4\pi}}\int_{s=0}^{t}\frac{1}{\sqrt{t-s}}\bigg(\int_{u=0}^{s}\bigg[-\frac{e^{-(x-\eta)^2/[4\alpha(s-u)]}}{k\sqrt{s-u}}q''(\eta,u) \\ &+\frac{e^{-x^2/[4\alpha(s-u)]}}{k\sqrt{s-u}}q''(0,u)-\frac{(x-\eta)}{2\alpha}\frac{e^{-(x-\eta)^2/[4\alpha(s-u)]}}{(s-u)^{\frac{3}{2}}}T(\eta,u) \\ &+\frac{x}{2\alpha}\frac{e^{-x^2/[4\alpha(s-u)]}}{(s-u)^{\frac{3}{2}}}T(0,u)\bigg]\mathrm{d}u\bigg)\mathrm{d}s \\ &+\sqrt{\frac{\alpha}{4\pi}}\int_{s=0}^{t}\frac{1}{\sqrt{t-s}}\bigg(\int_{u=0}^{s}\int_{x_o=0}^{\eta}\frac{e^{-(x-x_o)^2/[4\alpha(s-u)]}}{\sqrt{s-u}} \\ &\times F[T(x_o,u)]\,\mathrm{d}x_o\,\mathrm{d}u\bigg)\mathrm{d}s \qquad x\in(0,\eta), \qquad t\ge 0 \end{aligned} \tag{8}$$

Carefully interchanging orders of integration, on the triangle, permits the reduction of Eq. (8) to

$$\begin{aligned} &\int_{s=0}^{t}\frac{T(x,s)}{\sqrt{t-s}}\,\mathrm{d}s \\ &=\sqrt{\frac{\alpha}{4\pi}}\bigg[-\int_{u=0}^{t}\frac{q''(\eta,u)}{k}\int_{s=u}^{t}\frac{e^{-(x-\eta)^2/[4\alpha(s-u)]}}{\sqrt{s-u}\sqrt{t-s}}\,\mathrm{d}s\,\mathrm{d}u \\ &+\int_{u=0}^{t}\frac{q''(0,u)}{k}\int_{s=u}^{t}\frac{e^{-x^2/[4\alpha(s-u)]}}{\sqrt{s-u}\sqrt{t-s}}\,\mathrm{d}s\,\mathrm{d}u \\ &-\frac{(x-\eta)}{2\alpha}\int_{u=0}^{t}T(\eta,u)\int_{s=u}^{t}\frac{e^{-(x-\eta)^2/[4\alpha(s-u)]}}{(s-u)^{\frac{3}{2}}\sqrt{t-s}}\,\mathrm{d}s\,\mathrm{d}u \\ &+\frac{x}{2\alpha}\int_{u=0}^{t}T(0,u)\int_{s=u}^{t}\frac{e^{-x^2/[4\alpha(s-u)]}}{(s-u)^{\frac{3}{2}}\sqrt{t-s}}\,\mathrm{d}s\,\mathrm{d}u\bigg] \\ &+\sqrt{\frac{\alpha}{4\pi}}\int_{u=0}^{t}\int_{x_o=0}^{\eta}F[T(x_o,u)] \\ &\times\int_{s=u}^{t}\frac{e^{-(x-x_o)^2/[4\alpha(s-u)]}}{\sqrt{s-u}\sqrt{t-s}}\,\mathrm{d}s\,\mathrm{d}x_o\,\mathrm{d}u \qquad x\in(0,\eta), \qquad t\ge 0 \end{aligned} \tag{9a}$$

or

$$\begin{aligned} \int_{s=0}^{t}\frac{T(x,s)}{\sqrt{t-s}}\,\mathrm{d}s = &-\sqrt{\frac{\alpha\pi}{4k^2}}\int_{u=0}^{t}q''(\eta,u)\left(1-\operatorname{erf}\sqrt{\frac{(x-\eta)^2}{4\alpha(t-u)}}\right)\mathrm{d}u \\ &+\sqrt{\frac{\alpha\pi}{4k^2}}\int_{u=0}^{t}q''(0,u)\left(1-\operatorname{erf}\sqrt{\frac{x^2}{4\alpha(t-u)}}\right)\mathrm{d}u \\ &+\frac{1}{2}\int_{u=0}^{t}T(\eta,u)\frac{e^{-(x-\eta)^2/[4\alpha(t-u)]}}{\sqrt{t-u}}\mathrm{d}u \\ &+\frac{1}{2}\int_{u=0}^{t}T(0,u)\frac{e^{-x^2/[4\alpha(t-u)]}}{\sqrt{t-u}}\,\mathrm{d}u \\ &+\sqrt{\frac{\alpha\pi}{4}}\int_{u=0}^{t}\int_{x_o=0}^{\eta}F[T(x_o,u)]\left(1-\operatorname{erf}\sqrt{\frac{(x-x_o)^2}{4\alpha(t-u)}}\right)\mathrm{d}x_o\,\mathrm{d}u, \\ &x\in(0,\eta), \qquad t\ge 0 \end{aligned} \tag{9b}$$

where $\operatorname{erf}(z)$ is the error function with argument z [note: $\operatorname{erf}(0)=0$, $\operatorname{erf}(\infty)=1$, $\operatorname{erf}(z)=1-\operatorname{erfc}(z)$, where $\operatorname{erfc}(z)$ is the complementary error function] and where we made use of

$$\int_{s=u}^{t}\frac{e^{-a/(s-u)}}{\sqrt{t-s}\sqrt{s-u}}\,\mathrm{d}s = \pi\left[1-\operatorname{erf}\left(\sqrt{\frac{a}{(t-u)}}\right)\right] \tag{9c}$$

$$\int_{s=u}^{t}\frac{e^{-a/(s-u)}}{\sqrt{t-s}(s-u)^{\frac{3}{2}}}\mathrm{d}s = \sqrt{\frac{\pi}{a}}\frac{e^{-a/(t-u)}}{\sqrt{t-u}} \tag{9d}$$

Next, we let $s \to u$ for notational balance (i.e., cosmetics) integrate the LHS of Eq. (9b) by parts and incorporate the trivial initial condition to obtain

$$2\int_{u=0}^{t}\sqrt{t-u}\frac{\partial T}{\partial u}(x,u)\,\mathrm{d}u = -\sqrt{\frac{\alpha\pi}{4k^2}}\int_{u=0}^{t}q''(\eta,u) \times\left(1-\mathrm{erf}\sqrt{\frac{(x-\eta)^2}{4\alpha(t-u)}}\right)\mathrm{d}u + \sqrt{\frac{\alpha\pi}{4k^2}}\int_{u=0}^{t}q''(0,u) \times\left(1-\mathrm{erf}\sqrt{\frac{x^2}{4\alpha(t-u)}}\right)\mathrm{d}u + \frac{1}{2}\int_{u=0}^{t}T(\eta,u)\frac{e^{-(x-\eta)^2/[4\alpha(t-u)]}}{\sqrt{t-u}}\mathrm{d}u + \frac{1}{2}\int_{u=0}^{t}T(0,u)\frac{e^{-x^2/[4\alpha(t-u)]}}{\sqrt{t-u}}\mathrm{d}u + \sqrt{\frac{\alpha\pi}{4}}\int_{u=0}^{t}\int_{x_o=0}^{\eta}F[T(x_o,u)] \times\left(1-\mathrm{erf}\sqrt{\frac{(x-x_o)^2}{4\alpha(t-u)}}\right)\mathrm{d}x_o\,\mathrm{d}u,\quad x\in(0,\eta),\quad t\geq 0 \tag{10}$$

and then differentiate this result with respect to time using Leibnitz's rule to obtain

$$\int_{u=0}^{t}\frac{\partial T}{\partial u}(x,u)\frac{\mathrm{d}u}{\sqrt{t-u}} = -\frac{|x-\eta|}{4k}\int_{u=0}^{t}q''(\eta,u)\frac{e^{-(x-\eta)^2/[4\alpha(t-u)]}}{(t-u)^{\frac{3}{2}}}\mathrm{d}u + \frac{x}{4k}\int_{u=0}^{t}q''(0,u)\frac{e^{-x^2/[4\alpha(t-u)]}}{(t-u)^{\frac{3}{2}}}\mathrm{d}u + \frac{1}{2}\int_{u=0}^{t}T(\eta,u) \times\left[\frac{(x-\eta)^2}{4\alpha}\frac{e^{-(x-\eta)^2/[4\alpha(t-u)]}}{(t-u)^{\frac{5}{2}}} - \frac{1}{2}\frac{e^{-(x-\eta)^2/[4\alpha(t-u)]}}{(t-u)^{\frac{3}{2}}}\right]\mathrm{d}u + \frac{1}{2}\int_{u=0}^{t}T(0,u)\left[\frac{x^2}{4\alpha}\frac{e^{-x^2/[4\alpha(t-u)]}}{(t-u)^{\frac{5}{2}}} - \frac{1}{2}\frac{e^{-x^2/[4\alpha(t-u)]}}{(t-u)^{\frac{3}{2}}}\right]\mathrm{d}u + \frac{1}{4}\int_{u=0}^{t}\int_{x_o=0}^{\eta}F[T(x_o,u)]|x-x_o|\frac{e^{-(x-x_o)^2/[4\alpha(t-u)]}}{(t-u)^{\frac{3}{2}}}\mathrm{d}x_o\,\mathrm{d}u, \quad x\in(0,\eta),\qquad t\geq 0 \tag{11a}$$

where we have made use of

$$\frac{\partial}{\partial t}\mathrm{erfc}\left(\sqrt{\frac{a}{t-u}}\right) = \sqrt{\frac{a}{\pi}}\frac{e^{-a/(t-u)}}{(t-u)^{\frac{3}{2}}} \tag{11b}$$

$$\frac{\partial}{\partial t}\left(\frac{e^{-a/(t-u)}}{\sqrt{t-u}}\right) = \frac{ae^{-a/(t-u)}}{(t-u)^{\frac{5}{2}}} - \frac{1}{2}\frac{e^{-a/(t-u)}}{(t-u)^{\frac{3}{2}}} \tag{11c}$$

for various choices of a. Multiplying Eq. (7) by $\sqrt{\alpha\pi}/k$ produces an identifiable portion of the RHS of Eq. (11a). That is, we can eliminate the integral terms containing temperature and heat flux at $x=0$. Some care (caution) must be adhered to when simplifying the source term. Thus upon combining and recalling the definitions of the various material properties, we obtain the powerful relationship

$$q''(x,t) = \frac{k}{\sqrt{\alpha\pi}}\left[\int_{u=0}^{t}\frac{\partial T}{\partial u}(x,u)\frac{\mathrm{d}u}{\sqrt{t-u}} - \int_{u=0}^{t}T(\eta,u) \times\left(\frac{(x-\eta)^2}{4\alpha}\frac{e^{-(x-\eta)^2/[4\alpha(t-u)]}}{(t-u)^{\frac{5}{2}}} - \frac{1}{2}\frac{e^{-(x-\eta)^2/[4\alpha(t-u)]}}{(t-u)^{\frac{3}{2}}}\right)\mathrm{d}u + \frac{\eta-x}{2k}\int_{u=0}^{t}q''(\eta,u)\frac{e^{-(x-\eta)^2/[4\alpha(t-u)]}}{(t-u)^{\frac{3}{2}}}\mathrm{d}u\right] + \frac{k}{2\sqrt{\alpha\pi}}\int_{u=0}^{t}\int_{x_o=x}^{\eta}F[T(x_o,u)](x-x_o)\frac{e^{-(x-x_o)^2/[4\alpha(t-u)]}}{(t-u)^{\frac{3}{2}}}\mathrm{d}x_o\,\mathrm{d}u, \quad x\in(0,\eta),\quad t\geq 0 \tag{12}$$

Observe that in the limit as $x \to 0$ Eq. (12) replicates Eq. (7) [where $w_x = 1/2$] with the understanding that finite-part integration [27–31] is required for reexpressing the heating/cooling rate in terms of temperature.

The next section numerically verifies Eq. (12) by comparing the exact analytic solution of Eq. (2a), from two examples, to the numerically obtained results from Eq. (12) using a basic numerical method.

Numerical Validation

To numerically demonstrate Eq. (12), consider the following problem statement. Consider the heat equation described in Eq. (2a) subject to the trivial initial condition shown in Eq. (2b). Let the general boundary conditions for the slab geometry be given as

$$a_1\frac{\partial T}{\partial x}(0,t) + b_1T(0,t) = b_1T_1 \tag{13a}$$

$$a_2\frac{\partial T}{\partial x}(\eta,t) + b_2T(\eta,t) = b_2T_2 \tag{13b}$$

Under various conditions, an exact solution can be obtained by classical means without difficulty. For sake of simplicity, the material properties are chosen for copper as $\alpha = 0.00011\ \mathrm{m^2/s}$, $k = 400\ \mathrm{W/(m°C)}$. The slab thickness is chosen as $\eta = b = 0.01$ m. Consider case A in which $F[T(x,t)] = 0$, $b_1 = b_2 = 1$, $a_1 = a_2 = 0$, and $T_1 = T_2 = 200°\mathrm{C}$. This defines a symmetric problem with no volumetric heating. Consider case B in which $F[T(x,t)] = f_{\max} = 5\times 10^{6}{}°\mathrm{C/m^2}$, $a_1 = b_2 = 0$, $b_1 = a_2 = 1$, and $T_1 = 200°\mathrm{C}$. This defines a nonsymmetric case, in $x \in (0,\eta)$, with a constant volumetric source. The exact solutions (infinite series) can be obtained without recourse and are indicated by solid lines in the figures. The solid dots are results using Eq. (12) for estimating the heat flux based on left-hand rectangular rule integration assuming that the local heating/cooling rate is available, the back surface temperature at $x = \eta = b$ is known, and the heating source, $f_{\max}$ is provided.

Figure 1 displays case A results for the local heat flux at $x = b/2$ (line of symmetry) and $x = b/8$ (near the front surface) in the indicated time scale. Using only a simple numerical integration rule with the indicated number of points permits a rapid and accurate reconstruction of the local heat flux when compared with the exact solution. This verifies a portion of Eq. (12).

Figure 2 displays case B results for the local heat flux at $x = b/4$ and $x = b/8$ (near the front surface) in the indicated time scale. In this simulation, the source is present and the problem is nonsymmetrical in $x \in (0,b)$. Again, the numerical results for the local heat flux matches the exact solution. This is not a numerical study and thus convergence and error analysis are not present. These examples serve to validate Eq. (12) to the reader.

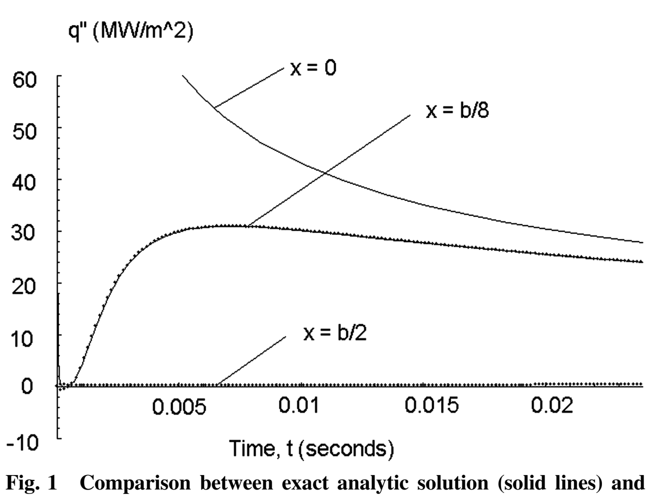

Fig. 1 Comparison between exact analytic solution (solid lines) and heat flux (dots), Eq. (12) for case A.

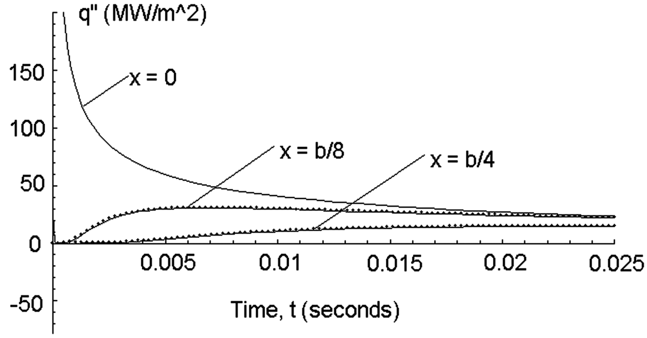

Fig. 2 Comparison between exact analytic solution (solid lines) and heat flux (dots), Eq. (12) for case B.

Conclusions

The integral relationship between temperature and heat flux leads to a transient means for determining the local heat flux in a finite slab containing a heating source. The local heat flux is obtained from either filtered temperature data [5,8] or directly from a heating/cooling rate sensor. The latter can lead to nearly real-time predictions of the heat flux. Digital filtering can be placed into a running form that could also be used in real-time applications. Frankel and Arimilli [5] and Frankel [8] described the Gaussian filtering scheme that incorporates a cutoff frequency determined from the power spectra of the signal. Thus, consideration of the signal-to-noise ratio is physically introduced for diffusive processes.

Acknowledgement

The author wishes to thank the observations and comments of an anonymous reviewer. This paper greatly benefited from his/her comments.

References

[1] Frankel, J. I., and Keyhani, M., "Inverse Heat Conduction: The Need of $\partial T/\partial t$ Data for Design and Diagnostic Purposes," *Proceedings of International Conference on Modeling, Identification and Controls, International Association of Science and Technology and Development*, 1999.

[2] Frankel, J. I., "Inverse Heat Conduction and Data-Type Issues," *Boundary Elements Communications*, Vol. 11, No. 4, 2000, pp. 37–42.

[3] Frankel, J. I., Osborne, G. E., and Taira, K., "Stabilization of Ill-Posed Problems Through Thermal Rate Sensors," *Journal of Thermophysics and Heat Transfer*, Vol. 20, No. 2, 2006, pp. 238–246.

[4] Frankel, J. I., and Lawless, J. J., "Numerically Stabilizing Ill-Posed Moving Surface Problems Through Heat Rate Sensors," *Journal of Thermophysics and Heat Transfer*, Vol. 19, No. 4, 2005, pp. 587–592.

[5] Frankel, J. I., and Arimilli, R. V., "Inferring Convective and Radiative Heating Loads from Transient Surface Temperature Measurements in the Half-Space," *Inverse Problems in Science and Engineering* (to be published).

[6] Frankel, J. I., and Osborne, G. E., "Motivation for the Development of Heating/Cooling Rate $\mathrm{d}T/\mathrm{d}t$ and Heat Flux Rate $dq''/\mathrm{d}t$ Sensors for Engineering Applications," *Proceedings of the 42nd AIAA Aerospace Sciences Meeting and Exhibit*, AIAA, Reston, VA, 2004.

[7] Frankel, J. I.,"Heating/Cooling Rate Sensor Development for Stable, Real-Time Heat Flux Predictions," *Proceedings of the 43rd AIAA Aerospace Sciences Meeting and Exhibit*, AIAA, Reston, VA, 2005.

[8] Frankel, J. I., Arimilli, R. V., Keyhani, M., and Wu, J., "Heating Rate $\mathrm{d}T/\mathrm{d}t$ Measurements Developed from In-Situ Thermocouples Using a Voltage-Rate Interface for Advanced Thermal Diagnostics," AIAA Paper 2006-3636.

[9] Kulish, V. V., Lage, J. L., Komarov, P. L., and Raad, P. E., "A Fractional-Diffusion Theory for Calculating Thermal Properties of Thin Films From Surface Transient Thermoreflectance Measurements," *Journal of Heat Transfer*, Vol. 123, No. 6, 2001, pp. 1133–1138.

[10] Kulish, V. V., and Novozhilov, V. B., "Integral Equation for the Heat Transfer with the Moving Boundary," *Journal of Thermophysics and Heat Transfer*, Vol. 17, No. 4, 2003, pp. 538–540.

[11] Kulish, V. V., and Lage, J. L., "Fractional Diffusion Solutions for Transient Local Temperature and Heat Flux," *Journal of Heat Transfer*, Vol. 122, No. 2, 2000, pp. 372–376.

[12] Kress, R., *Linear Integral Equations*, Springer–Verlag, Berlin, 1989.

[13] Wing, G. M., *A Primer on Integral Equations the First Kind*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1991.

[14] Linz, P., *Analytical and Numerical Methods for Volterra Equations*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1985.

[15] Baker, C. T. H., *The Numerical Treatment of Integral Equations*, Clarendon Press, Oxford, 1978.

[16] Hanke, M., and Scherzer, O., "Inverse Problems Light: Numerical Differentiation," *American Mathematical Monthly*, Vol. 108, No. 6, 2001, pp. 512–520.

[17] Groetsch, T., "Differentiation of Approximately Specified Functions," *American Mathematical Monthly*, Vol. 98, No. 9, 1991, pp. 847–850.

[18] Doebelin, E. O., *Measurement Systems*, McGraw–Hill, New York, 1975.

[19] Diller, T. E., "Advances in Heat Flux Measurements," *Advances in Heat Transfer*, Vol. 23, Academic Press, New York, 1993, pp. 279–368.

[20] Keltner, N. R., "Heat Flux Measurements: Theory and Applications," *Thermal Measurements in Electronic Cooling*, CRC, New York, 1997, pp. 273–320.

[21] Holman, J. P., *Experimental Methods for Engineers*, 4th ed., McGraw–Hill, New York, 1984.

[22] Ozisik, M. N., *Heat Conduction*, Wiley, New York, 1980.

[23] Greenberg, M. D., *Applications of Green's Functions in Science and Engineering*, Prentice–Hall, Englewoood Cliffs, New Jersey, 1971, p. 100.

[24] Frankel, J. I., Vick, B., and Ozisik, M. N., "Flux Formulation in Hyperbolic Heat Conduction," *Journal of Applied Physics*, Vol. 58, No. 9, 1985, pp. 3340–3345.

[25] Stakgold, I., *Green's Functions and Boundary Value Problems*, Wiley, New York, 1979.

[26] Brebbia, C. A., Telles, J. C. F., and Wrobel, L. C., *Boundary Element Techniques*, Springer–Verlag, New York, 1984.

[27] Kythe, P. K., and Puri, P., *Computational Methods for Linear Integral Equations*, Birkhauser, Boston, MA, 2002.

[28] Martin, P. A., "Exact Solution of a Simple Hypersingular Integral Equation," *Journal of Integral Equations and Applications*, Vol. 4, No. 2, 1992, pp. 197–204.

[29] Kaya, A. C., and Erdogan, F., "On the Solution of Integral Equations with Strongly Singular Kernels," *Quarterly of Applied Mathematics*, Vol. 45, No. 1, 1987, pp. 105–122.

[30] Monegato, G., "Numerical Evaluation of Hypersingular Integrals," *Journal of Computational and Applied Mathematics*, Vol. 50, Nos. 1–3, 1994, pp. 9–31.

[31] Golberg, M. A. (ed.), *Solution Methods for Integral Equations*, Plenum, New York, 1979.